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Existence of Delaunay Pairwise Gibbs Point Process with Superstable Component

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The present stuffy deals with the existence of Delaunay pairwise Gibbs point process with superstable component by using the well-known Preston theorem. In particular, we prove the stability, the lower regularity, and the quasilocality properties of the Delaunay model.

KEY WORDS: Gibbs states; Delaunay triangulation; pairwise interaction; DLR equations; local specifications; superstable interaction.

1. INTRODUCTION

During the last few years significant advances took place in the statistical analysis of spatial point patterns as summarized in the book of Stoyan *et al.*[18] which mainly result from the development of the theory of such processes in a mathematical framework. The importance of the Gibbs point process as a model building principle became widely recognized through these studies. Indeed, the class of Gibbs point processes is interesting because it allows to introduce and study interactions between points through the modelling of an associated potential. This resulting gain explains their use in statistical physics.

Within the statistical mechanics framework, Gibbs states are defined as solutions of the well known equilibrium equations referred as the Dobrushin–Lanford–Ruelle (D.L.R.) equations.[4, 12] In particular, with the help of the useful correlation functions,[16] Ruelle established the equivalence between equilibrium equations and the Kirkwood–Salsburg equations for pairwise interactions model.[17] This result has been extended to higher order interactions.[9, 10] Another way to introduce Gibbs point

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process consist in using a familly of local specifications with respect to a weight process. The Preston's theorems[15] used precisely this approach in order to give sufficient conditions on local specifications for existence of Gibbs states. Moreover, applying one of these theorems, Klein proposed convenient conditions which carry on local energy[11] and yield a large class of continuum many body potentials. In recent studies, the Gibbs variational formula of the pressure which is another characterisation of Gibbs states, is obtained by the large deviation techniques for superstable regular Ruelle's class[5] and continuous Potts models.[6]

In 1989, Baddeley and Møller made some connection between Gibbs point processes and Delaunay graph with the well known "nearest-neighbour" Markov point processes.[1]

Avoiding a possible hard-core condition considered in ref. 3, we proved the existence of some Delaunay Gibbs models.[2] A condition on the smallest angle of Delaunay triangles gives a lower bound for the local energy. Moreover, assuming that the interaction potential vanishes when the radius of the circle circumscribed by each Delaunay triangle tends to infinity, we obtain the quasilocality property. It allows us to use the Theorem 3.1 of Preston.[15] Within the same framework, we derive similar results for pairwise Delaunay model with the smallest angle criteria.

By relaxing the smallest angle condition, we propose here another way to prove the existence of Gibbs states using the Theorem 3.3 of Preston.[15] Since the Delaunay pairwise model is stable but not superstable we add a classical pairwise superstable component acting on the complete graph. Moreover, the lower regularity and quasilocality properties are obtained under some greatest angle condition, less restrictive than the smallest angle one, and a regularity assumption on the pairwise potential which controls the length of the residual edges.

Some simulations for Delaunay pairwise and triple interaction are proposed in refs. 1 and 3. Such simulations are based on the Geyer and Møller algorithm.[8, 7]

After some general preliminaries about Delaunay triangulation and point processes (Section 2), we give a presentation of Delaunay pairwise models and another one with a superstable component (Section 3). In Section 4, we establish the stability, the lower regularity and weak kind of quasilocality properties of the Delaunay model with greatest angle condition. Finally, we prove the existence of Gibbs state.

2. NOTATIONS AND PRELIMINARIES

First of all, let us introduce some general notations. $|A|$ denotes the Lebesgue measure, when the set A is a bounded Borel set of $\mathbb{R}^2$, and the counting measure if A is a discrete set. For any sets A and B, we define

$$A \vee B = \{z_1 \cup z_2 : z_1 \in A, z_2 \in B \text{ and } z_1 \neq z_2\}$$

Given any set A, $\mathscr{P}(A)$ and $\mathscr{P}_2(A) = A \vee A$ denote respectively the set of subsets of A and the set of subsets with two distinct elements of A.

2.1. Delaunay graph

Let ψ be a triangle, we denote: $\mathscr{C}(\psi)$ the circle circumscribed of ψ, $R(\psi)$ the radius of $\mathscr{C}(\psi)$, $h(\psi)$ the greatest edge of ψ, $\alpha(\psi)$ the greatest angle of ψ and $\beta(\psi)$ the smallest angle of ψ. We call configuration (of points) a locally finite subset of $\mathbb{R}^d$.

Definition 1. The Delaunay graph of some configuration φ in $\mathbb{R}^2$ is the unique triangulation in which the interior of the circle $C(\psi)$ circumscribed by every triangle ψ of the triangulation does not contain any point of φ (see Fig. 1): $C(\psi) \cap \varphi = \varnothing$.[14]

In fact, the definition supposes that the configuration φ is in a general position (four points on the same circle are not possible) in a way to ensure the existence and the uniqueness of the Delaunay graph.[13] In this work, we want to relax this assumption and we extend the previous definition by adopting any determistic rule allowing a choice of triangulation of a subconfiguration $\psi \subset \varphi$ of points located on a same circle (with no point

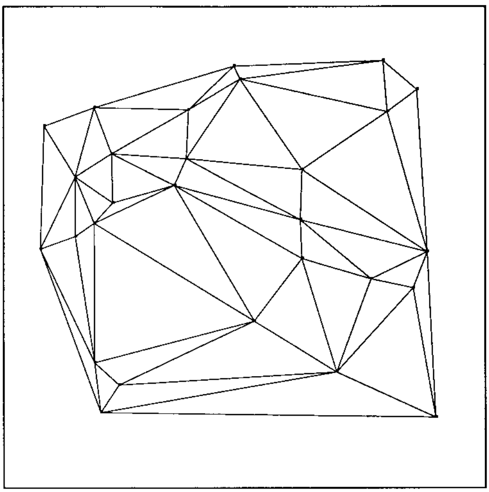

Fig. 1. Delaunay triangulation.

inside the circle). For instance, a rule based on the lexicographic order on the polar coordinates could be admissible.

Let $Del_2(\varphi)$ be the Delaunay edges and $Del_3(\varphi)$ be the Delaunay triangles of a configuration φ. Let us introduce the set $\mathcal{DC}(\varphi)$ defined by:

$$\mathcal{DC}(\varphi) = Del_2(\varphi) \cup Del_3(\varphi)$$

We point out that for any $\chi \subset \varphi$,

$$\chi \in \mathcal{DC}(\varphi \cup \psi) \Rightarrow \chi \in \mathcal{DC}(\varphi) \tag{1}$$

Let φ be a configuration of points and $x \in \varphi$. The neighbourhood of x in φ is:

$$N(x \mid \varphi) = \{x_i \in \varphi, \{x, x_i\} \in Del_2(\varphi)\}$$

Let x, y_1, y_2 be three points of any configuration φ such that $\{y_1, y_2\} \in Del_2(\varphi)$. We denote: $[y_1, y_2]$ and $]y_1, y_2[$ the closed and open segments between y_1 and y_2, D_{y_1, y_2} the straight line crossing y_1 and y_2, $\Pi^{ext}(x, \{y_1, y_2\})$ the half-plane not containing the point x where the boundary is supported by the straight line D_{y_1, y_2} and $\gamma(x, \{y_1, y_2\})$ the angle at x of the triangle $\{x, y_1, y_2\}$. Furthermore, we let:

$$\mathcal{C}^{ext}(x, \{y_1, y_2\}) = \mathcal{C}(\{x, y_1, y_2\}) \cap \Pi^{ext}(x, \{y_1, y_2\})$$
$$X^{opp}(\{y_1, y_2\}, \varphi) = \{x \in \varphi: \{x, y_1, y_2\} \in Del_3(\varphi)\}$$

Before defining particular subgraphs of the Delaunay graph $Del_2(\varphi)$, we introduce the following particular subsets of $Del_3(\varphi)$:

$$Del^{\beta_0}_{3,\beta}(\varphi) = \{\psi \in Del_3(\varphi): \beta(\psi) > \beta_0\}$$
$$Del^{\alpha_0}_{3,\alpha}(\varphi) = \{\psi \in Del_3(\varphi): \alpha(\psi) < \alpha_0\}$$

Definition 2. 1. Given any $\beta_0 \in]0, \pi/3]$, the β-Delaunay graph of order β_0 of any configuration φ is the Delaunay subgraph defined by:

$$Del^{\beta_0}_{2,\beta}(\varphi) = \bigcup_{\psi \in Del^{\beta_0}_{3,\beta}(\varphi)} \{\xi \subset \psi: |\xi| = 2\}$$

2. Given any $\alpha_0 \in [\pi/2, \pi[$, the α-Delaunay graph of order α_0 of any configuration φ is the Delaunay subgraph defined by:

$$\begin{aligned} Del^{\alpha_0}_{2,\alpha}(\varphi) &= \{\xi \in Del_2(\varphi): \forall x \in X^{opp}(\xi, \varphi), \gamma(x, \xi) < \alpha_0\} \\ &= Del_2(\varphi) \backslash h(Del_3(\varphi) \backslash Del^{\alpha_0}_{3,\alpha}(\varphi)) \end{aligned} \tag{2}$$

The α-Delaunay graph of order $\pi/2$ is the well-known Gabriel graph. One may extend the definition of $Del_{2,\alpha}^{\alpha_0}(\varphi)$ for $\alpha_0 \in]0, \pi/2[$ by using (2) only.

In particular, we have for any given $\alpha_0 \geqslant \pi/3$ and $\beta_0 \geqslant (\pi - \alpha_0)/2$:

$$Del_{2,\beta}^{\beta_0}(\varphi) \subset Del_{2,\alpha}^{\alpha_0}(\varphi) \subset Del_2(\varphi) \tag{3}$$

In the rest of the paper we will need the following notation:

$$\overline{Del_{2,\alpha}^{\alpha_0}(\varphi)} = \mathscr{P}_2(\varphi) \setminus Del_{2,\alpha}^{\alpha_0}(\varphi)$$

The following result is a direct consequence of the elementary property of inscribed angles.

Lemma 1. Let $z_1, z_2 \in X^{opp}(\{y_1, y_2\}, \varphi)$.

1. If $\gamma(z_1, \{y_1, y_2\}) \in [0, \pi/2[$ and $\gamma(z_2, \{y_1, y_2\}) \in [0, \pi/2[$ then

$$\|y_1 - y_2\| \geqslant \|x_1 - x_2\| (\geqslant \sin(\alpha_0) \|x_1 - x_2\|),$$

$$\forall x_1, x_2 \in \bigcup_{i=1,2} \mathscr{C}^{ext}(z_i, \{y_1, y_2\})$$

2. If there exists $i \in \{1, 2\}$ such that $\gamma(z_i, \{y_1, y_2\}) \in [\pi/2, \alpha_0[$ then

$$\|y_1 - y_2\| \geqslant \sin(\alpha_0) \|x_1 - x_2\|, \qquad \forall x_1, x_2 \in \mathscr{C}(y_1, z_i, y_2)$$

2.2. Gibbs Point Processes

We denote by $\mathscr{B}$ the Borel σ-field, $\mathscr{B}_b$ the bounded Borel boolean ring of $\mathbb{R}^d$, and $\mathscr{K}$ the set of compact subsets of $\mathbb{R}^d$. Let Ω denotes the class of all configurations. In particular, an element φ of Ω could be represented by $\varphi = \sum_{i \in \mathbb{N}} \delta_{x_i}$ which is a simple counting Radon measure in $\mathbb{R}^d$ (i.e., all the points x_i of $\mathbb{R}^d$ are distinct) where for all $\Lambda \in \mathscr{B}$, $\delta_x(\Lambda) = 1_\Lambda(x)$ is the Dirac measure and $1_A(\cdot)$ is the indicator function of a set A. This space Ω is equipped with the vague topology, where the weak topology for Radon measures with respect to the set of continuous functions vanishing outside a compact set. $\mathscr{F}$ is the σ-field spanned by the maps $\varphi \to \varphi(\Lambda)$, $\Lambda \in \mathscr{B}_b$. The set of all configurations in a measurable set $\Lambda \subset \mathbb{R}^d$ will be denoted by Ω_Λ and the corresponding σ-field $\mathscr{F}_\Lambda$ is similarly defined. Furthermore, for any $\Lambda \in \mathscr{B}_b$,

$$(\Omega, \mathscr{F}) = (\Omega_\Lambda, \mathscr{F}_\Lambda) \times (\Omega_{\Lambda^c}, \mathscr{F}_{\Lambda^c})$$

where Λ^c denotes the complement of Λ in $\mathbb{R}^d$. Let $\tilde{\mathscr{F}}_\Lambda$ be the reverse projection of $\mathscr{F}_\Lambda$ under the previous identification, so that $\tilde{\mathscr{F}}_\Lambda$ is a σ-field on Ω. Finally, Ω_f denotes the class of all finite subsets of $\mathbb{R}^d$.

The local energy $E: \Omega_f \times \Omega \to \mathbb{R} \cup \{+\infty\}$ is a measurable function such that, for any $\varphi \in \Omega$ and any permutation σ of any n distinct points $x_1,..., x_n \in \mathbb{R}^d$:

$$E\left(\sum_{i=1}^{n} \delta_{x_i}, \varphi\right) = E(x_{\sigma(1)}, \varphi) + E(x_{\sigma(2)}, \varphi + \delta_{x_{\sigma(1)}}) + \cdots + E\left(x_{\sigma(n)}, \varphi + \sum_{i=1}^{n-1} \delta_{x_{\sigma(i)}}\right)$$

This last quantity is physically interpreted as the energy required to add the points $x_1,..., x_n$ into the configuration φ. Furthermore,

$$V(\varphi) = E(\varphi, \varnothing)$$

is physically interpreted as the finite energy of the configuration φ adopting the convention $V(\varnothing) = 0$. Under the finiteness of $V(\varphi \cup \psi)$, the mutual energy $W(\varphi, \psi)$ between the two finite configurations φ and ψ is defined by:

$$W(\varphi, \psi) = V(\varphi \cup \psi) - V(\varphi) - V(\psi) = E(\varphi, \psi) - E(\varphi, \varnothing)$$

Later on, we adopt some notations similar to that of Preston and we define the same sets R^0_Λ, R^+_Λ, R^-_Λ and R_Λ used by Preston (p. 97, ref. 15). Given some configuration $\psi \in R_\Lambda$, we denote, for any $\varphi \in \Omega_\Lambda$:

$$V_\Lambda(\varphi, \psi) = E(\varphi, \psi_{\Lambda^c}) = V(\varphi \cup \psi_{\Lambda^c}) - V(\psi_{\Lambda^c}) \qquad \text{if} \quad \psi_{\Lambda^c} \in \Omega_f$$

and

$$V_\Lambda(\varphi, \psi) = \lim_{\tilde{\Lambda} \to \mathbf{R}^2} V^{\tilde{\Lambda}}_\Lambda(\varphi, \psi)$$

when the limit exists and where, for any $\tilde{\Lambda} \supset \Lambda$,

$$V^{\tilde{\Lambda}}_\Lambda(\varphi, \psi) = V_\Lambda(\varphi, \psi \cap \tilde{\Lambda})$$

A Gibbs point process on $\mathbb{R}^d$ is a probability distribution $\mathbf{P}$ on $(\Omega, \mathscr{F})$ and is usually defined using a family of local specifications with respect to a weight process (often a stationary Poisson process with distribution $\mathbf{Q}$

and intensity $\lambda_{\mathbf{Q}}=1$). For such a process, given some configuration ψ in Ω, the conditional probability on $\Lambda \in \mathscr{B}_b$ is of the form:

$$\pi_\Lambda(\psi, F) = \left\{ \frac{1}{Z_\Lambda(\psi)} \in \int_{\Omega_\Lambda} \exp(-V_\Lambda(\varphi, \psi))\, 1_F(\varphi \cup \psi_{\Lambda^c})\, \mathbf{Q}_\Lambda(d\varphi) \right\} 1_{R_\Lambda}(\psi)$$

for any $F \in \mathscr{F}$, where

$$Z_\Lambda(\psi) = \int_{\Omega_\Lambda} \exp(-V_\Lambda(\varphi, \psi))\, \mathbf{Q}_\Lambda(d\varphi)$$

is called the partition function and $\mathbf{Q}_\Lambda$ is the measure (corresponding to the Poisson process with intensity λ) expressed by:

$$\mathbf{Q}_\Lambda(d\varphi) = \exp(-\lambda|\Lambda|) \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} d^n\varphi$$

It is well known that the collection of probability kernels $(\pi_\Lambda)_{\Lambda \in \mathscr{B}_b}$ satisfies the set of compatibility and measurability conditions which defines a local specification in the Preston's sense. The main condition is that of consistency:

$$\pi_\Lambda \pi_{\Lambda'} = \pi_\Lambda \qquad \text{for any} \quad \Lambda' \subset \Lambda$$

Let us recall the Theorem 3.3 of Preston (p. 42, ref. 15) which is often used to prove the existence of a Gibbs state. Here, we use the condition numbers of this theorem.

Definition 3. We call a cylindric function, a real-valued function on Ω such that:

$$\forall \varphi \in \Omega, \qquad \exists D \in \mathscr{B}_b\colon f(\varphi) = f(\varphi \cap D)$$

A set A is said to be cylindric if its indicator function 1_A is a cylindric function.

Let C_k be the hypercube in $\mathbb{R}^d$ of length $2k$ centered at the origin. We introduce the following two sets:

$$\mathscr{H} = \{\varphi \in \Omega : \forall \psi \subset \varphi,\ \psi \in \Omega_f,\ V(\psi) < +\infty\}$$

$$U_m = \mathscr{H} \cap \bigcap_{k \geqslant 1} \{\varphi \in \Omega,\ \varphi(C_k \backslash C_{k-1}) \leqslant m|C_k \backslash C_{k-1}|\}$$

Theorem 1. Let $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathscr{B}_b}$ be a local specification with respect to $R = \{R_\Lambda\}_{\Lambda \in \mathscr{B}_b}$. Assume that:

(3.7) Given $\Theta \in \mathscr{K}$ and $\gamma > 0$, there exists a probability measure ω on $\tilde{\mathscr{F}}_\Theta$, $\Lambda \in \mathscr{B}_b$ and $\delta > 0$ such that:

$$(F \in \tilde{\mathscr{F}}_\Theta \text{ with } \omega(F) < \delta) \Rightarrow (\pi_{\tilde{\Lambda}}(\emptyset, F) < \gamma, \forall \tilde{\Lambda} \supset \Lambda)$$

(3.12) Given any $\delta > 0$, there exists $m \geqslant 1$ such that:

$$\pi_\Lambda(\emptyset, U_m) > 1 - \delta, \qquad \forall \Lambda \in \mathscr{B}_b$$

(3.13) Given $\Lambda \in \mathscr{B}_b$, $\gamma > 0$ and a cylindric set $F \in \mathscr{F}$, there exists a cylindric function f such that:

$$|\pi_\Lambda(\varphi, F) - f(\varphi)| < \gamma \qquad \forall \varphi \in U_m$$

Then $\mathscr{G}(E) \neq \emptyset$, where $\mathscr{G}$ is the set of Gibbs measures $\mathbf{P}$, associated with the local specification Π, satisfying the D.L.R equations.

Denoting by φ and ψ two finite configurations, we define a unit cube

$$\mathscr{Q}(r) = \{x = (x^1,..., x^d) \in \mathbb{R}^d \colon \forall i \in \{1,..., d\}, r^i - \tfrac{1}{2} \leqslant x^i \leqslant r^i + \tfrac{1}{2}\}$$

The finite subsets $\mathscr{R}$ and $\mathscr{S}$ of $\mathbb{Z}^d$ are defined such that $\varphi \in \bigcup_{r \in \mathscr{R}} \mathscr{Q}(r)$ and $\psi \in \bigcup_{s \in \mathscr{S}} \mathscr{Q}(s)$. Besides, if $r \in \mathbb{Z}^d$, we let $|r| = \sup_{i \in \{1,..., d\}} |r^i|$.

Let us now recall the well known assumptions of superstability and regularity of ref. 17:

- *Superstability*: there exists $A > 0$ and $B \geqslant 0$ such that

$$V(\varphi) > \sum_{r \in \mathscr{R}} (An^2(\varphi, r) - Bn(\varphi, r))$$

where $n(\varphi, r)$ is the number of points of the configuration φ in the hypercube $\mathscr{Q}(r)$.

- *Lower Regularity*:

$$W(\varphi, \psi) > -\sum_{r \in \mathscr{R}} \sum_{s \in \mathscr{S}} \Psi(|r - s|) \frac{(n^2(\psi, s) + n^2(\varphi, r))}{2}$$

where Ψ is a decreasing function satisfying $\sum_{r \in \mathbb{Z}^d} \Psi(|r|) < +\infty$.

In the following, we restrict our study to point processes in $\mathbb{R}^2$.

3. PRESENTATION OF THE MODEL

Due to the linear complexity of the Delaunay graph, the stability property for Delaunay pairwise model is obtained by choosing a lower bounded pairwise potential. The superstability property cannot be obtained using the same argument. The classical approach being impossible, we adopt here two different ones.

The first approach consists of considering Delaunay subgraphs such that models based on these subgraphs are local stable and quasilocal. In particular such a subgraph, proposed in this paper and named β-Delaunay graph, is obtained by keeping all edges of the Delaunay triangles with the smallest angle greater than a fixed value β_0. Thus, the Theorem 3.1 of Preston[15] leads to the existence of Gibbs state for this model. One should notice that this approach is very similar to the one adopted in a previous paper.[2]

In the second approach, we are interested in the Delaunay subgraph, called α-Delaunay graph, obtained by deleting the greatest edge of every triangles with greatest angle greater than a fixed value α_0. The values of α_0 and β_0 can be chosen such that the α-Delaunay graph contains the β-Delaunay graph. The α-Delaunay pairwise model satisfies the local stability and quasilocality properties if a hard-core condition is assumed on the pairwise potential. When this condition is not satisfied, we may introduce a new model by the addition of a superstable component. Under some suitable integrability assumption of the α-Delaunay pairwise potential at infinity, we show that the α-Delaunay pairwise model is lower regular and satisfies a weak kind of quasilocality property introduced in the Theorem 3.3 of Preston.[15] These two properties could not be obtained for the Delaunay pairwise model. Consequently, the α-Delaunay pairwise model with superstable component satisfies the hypothesis of Theorem 3.3 of Preston[15] yielding the existence of Gibbs state.

3.1. β-Delaunay Pairwise Model

In order to introduce the next model based on the α-Delaunay graph, we first deal with a model based on the β-Delaunay graph which is a subgraph of the previous one (see (3)). The finite energy is then expressed for any $\varphi \in \Omega_f$ as:

$$V(\varphi) = \sum_{\xi \in Del_{2,\beta}^{\beta_0}(\varphi)} \phi(\xi) \tag{4}$$

where ϕ is a pairwise potential satisfying some suitable assumptions given below. The existence of Gibbs state based on this finite energy is mainly derived from a previous paper[2] that deals with the existence of model with finite energy of the form:

$$V(\varphi) = \sum_{\psi \in Del^{\beta_0}_{3,\beta}(\varphi)} \phi(\psi) \tag{5}$$

Indeed, we can prove that for any configurations φ_1 and φ_2 we have:

$$Del^{\beta_0}_{2,\beta}(\varphi_1) \setminus Del^{\beta_0}_{2,\beta}(\varphi_2) \subset \{\xi \subset \psi : |\xi| = 2 \text{ and } \psi \in Del^{\beta_0}_{3,\beta}(\varphi_1) \setminus Del^{\beta_0}_{3,\beta}(\varphi_2)\}$$

in particular, this means that any residual edge of the local energy associated to finite energy given by (4) is contained in some residual triangle of the local energy associated to finite energy given by (5). In fact, the model studied in this subsection is very similar to the following model proposed in ref. 2 as a particular case of (5):

$$V(\varphi) = \frac{1}{2} \sum_{\psi \in Del^{\beta_0}_{3,\beta}(\varphi)} \sum_{\substack{\xi \subset \psi \\ |\xi| = 2}} \phi(\xi)$$

for which we assume that:

$$(\mathbf{H}): \quad \forall y_1, y_2 \in \mathbb{R}^2, \qquad |\phi(\{y_1, y_2\})| \leqslant \tilde{\phi}(\|y_1 - y_2\|) \leqslant K$$

with $\tilde{\phi}$ a positive decreasing function which vanishes asymptotically and K a positive constant.

Under (**H**), the existence of stationary Gibbs state is proved for the two models exactly in the same way (see ref. 2 for further details).

3.2. α-Delaunay Pairwise Model

The Delaunay graph is a kind of "multidirectional" nearest neighbours graph. In particular, two points cannot be neighbours if there exists one point aligned between them. When points represent particules of diameter $\varepsilon > 0$ one may think that two particules do not interact together if another one is quasi-aligned between them. Thus, the α-Delaunay graph chosen near enough from π, expresses this type of interaction and generalizes the concept of interactions in the Delaunay sense for three particules quasi-aligned.

Originally, our-goal was to study the existence of Gibbs state associated with pairwise finite energy based on the Delaunay graph. Unfortunately, this model is not quasilocal (and lower regular too) because of the contribution of residual edges which do not vanish asymptotically. One way to solve this problem is to introduce some greatest angle criteria and define the α-Delaunay graph.

Given $\varphi \in \Omega_f$, the finite energy is expressed as

$$V(\varphi) = \sum_{\psi \in Del^{\alpha_0}_{2,\alpha}(\varphi)} \phi(\xi) \tag{6}$$

where ϕ is a lower bounded function such that there exists a positive decreasing function $\tilde{\phi}$ satisfying for any points y_1 and y_2 in $\mathbb{R}^2$:

$$|\phi(\{y_1, y_2\})| \leqslant \tilde{\phi}(\|y_1 - y_2\|) \qquad \text{and} \qquad \int_0^{+\infty} t\tilde{\phi}(t)\, dt < +\infty$$

Generally, ϕ is of the form, $\phi(\xi) = g(\|y_1 - y_2\|)$, for any $\xi = \{y_1, y_2\} \in Del^{\alpha_0}_{2,\alpha}(\varphi)$, where g is some real valued function.

Remark 1. 1. When $\alpha_0 > \pi/2$, we may express the Delaunay pairwise model as:

$$V(\varphi) = \sum_{\psi \in \mathscr{DC}(\varphi)} \phi_{del}(\psi)$$

where, for any $\psi \in \mathscr{DC}(\varphi)$,

$$\phi_{del}(\psi) = \begin{cases} \phi(\psi) & \text{if} \quad \psi \in Del_2(\varphi) \\ -\phi(h(\psi)) & \text{if} \quad \psi \notin Del^{\alpha_0}_{3,\alpha}(\varphi) \\ 0 & \text{otherwise} \end{cases}$$

2. Given α_0 and R_0, this remains true when we replace $Del^{\alpha_0}_{3,\alpha}$ by:

$$Del^{\alpha_0, R_0}_{3,\alpha,R}(\varphi) = \{\psi \in Del_3(\varphi) : \alpha(\psi) < \alpha_0 \text{ or } R(\psi) < R_0\}$$

3. The condition on the greatest angle ($\alpha(\psi) > \alpha_0$) is required here in order to obtain the quasilocality and lower regularity properties.

3.3. Mixed Pairwise Model

In a previous paper[3] we introduced the Delaunay pairwise model with a hard-core component in a bounded Borel set. In this section, we replace the hard-core component by a more general superstable component.

Given $\varphi \in \Omega_f$, the finite energy is $V(\varphi) = V_1(\varphi) + V_2(\varphi)$ where $V_1(\varphi)$ is an α-Delaunay pairwise finite energy and $V_2(\varphi)$ is a superstable and lower regular finite energy. In particular, in the pairwise interaction case the local energy $V_2(\varphi)$ is:

$$V_2(\varphi) = \sum_{\{x, y\} \in \varphi, x \neq y} \phi_{sr}(x-y)$$

where ϕ_{sr} denotes the pairwise interaction potential satisfying the Dobrushin, Fisher and Ruelle assumptions:[16]

$$\begin{cases} \phi_{sr} \geqslant -K, \ (K \geqslant 0) & \\ \phi_{sr}(x) \geqslant \phi_1(|x|) & \text{for} \quad |x| \leqslant r_1 \\ |\phi_{sr}(x)| \leqslant \phi_2(|x|) & \text{for} \quad |x| \geqslant r_2 \end{cases}$$

For any $0 < r_1 < r_2 < +\infty$, ϕ_1 and ϕ_2 are positive decreasing functions such that

$$\begin{cases} \phi_1 \colon [0, r_1] \to \mathbb{R} \cup \{+\infty\} & \text{and} \quad \displaystyle\int_0^{r_1} t\phi_1(t)\, dt = +\infty \\ \phi_2 \colon [r_2, +\infty) \to \mathbb{R} & \text{and} \quad \displaystyle\int_{r_2}^{+\infty} t\phi_2(t)\, dt < +\infty \end{cases}$$

These assumptions are sufficient for the regularity and the superstability of ϕ_{sr}. The addition of the classical superstable component allows us to use a Delaunay pairwise interaction potential of the Lennard–Jones type.

4. PROPERTIES OF THE α-DELAUNAY PAIRWISE MODEL

In this section, we establish respectively the stability, the lower regularity and some weak kind of quasilocality properties of the α-Delaunay pairwise model.

4.1. Stability

Clearly the stability is a direct consequence of the linear complexity of the Delaunay graph (Corollary 5.2, p. 105 of ref. 14):

$$V(\varphi) \geqslant 3 \inf(\phi)\, |\varphi| \tag{7}$$

We point out that the stability for classical pairwise models may require more restrictive conditions on the interaction potential. Due to the linear-complexity of the Delaunay graph, we cannot obtain the superstability property.

4.2. Lower Regularity

We notice that for our model two kinds of contribution appear in the expression of $W(\varphi, \psi)$. The first one, called here "positive" contribution and denoted by $W_+(\varphi, \psi)$, corresponds to creating edges that connect a point of φ to a point of ψ. The second one, that is unexpected, is called here "negative" contribution and is denoted by $W_-(\varphi, \psi)$, and it corresponds to deleting edges (i.e., broken edges) that connect two points of φ or two points of ψ. The mutual energy between configurations φ and ψ can be written as:

$$W(\varphi, \psi) = W^+(\varphi, \psi) + W^-(\varphi, \psi)$$

We point out that in the classical models, there is only a "positive" contribution for the mutual energy (i.e., $W(\varphi, \psi) = W^+(\varphi, \psi)$).

Before expressing the mutual energy between two finite configurations φ and ψ, we introduce the particular sets of edges:

$$Del_2^+(\psi, \varphi) = Del_2(\psi \cup \varphi) \backslash Del_2(\varphi) \text{ and}$$

$$Del_{2,\alpha}^{\alpha_0,+}(\psi, \varphi) = Del_{2,\alpha}^{\alpha_0}(\psi \cup \varphi) \backslash Del_{2,\alpha}^{\alpha_0}(\varphi)$$

$$Del_2^-(\psi, \varphi) = Del_2(\varphi) \backslash Del_2(\psi \cup \varphi) \text{ and}$$

$$Del_{2,\alpha}^{\alpha_0,-}(\psi, \varphi) = Del_{2,\alpha}^{\alpha_0}(\varphi) \backslash Del_{2,\alpha}^{\alpha_0}(\psi \cup \varphi)$$

By convenience, when ψ reduces to a singleton $\{\omega\}$ we denote $Del_2^+(\psi, \varphi)$ (resp. $Del_{2,\alpha}^{\alpha_0,+}(\psi, \varphi)$, $Del_2^-(\psi, \varphi)$ and $Del_{2,\alpha}^{\alpha_0,-}(\psi, \varphi)$) by $Del_2^+(\omega, \varphi)$ (resp. $Del_{2,\alpha}^{\alpha_0,+}(\omega, \varphi)$, $Del_2^-(\omega, \varphi)$ and $Del_{2,\alpha}^{\alpha_0,-}(\omega, \varphi)$).

More precisely, in the α-Delaunay pairwise model we have

$$\begin{aligned} W^+(\varphi, \psi) &= \sum_{\xi \in Del_{2,\alpha}^{\alpha_0,+}(\psi, \varphi)} \phi(\xi) \\ &= \sum_{\substack{\{z, \omega\} \in Del_{2,\alpha}^{\alpha_0}(\varphi \cup \psi) \\ z \in \varphi,\, \omega \in \psi}} \phi(\{\omega, z\}) \end{aligned}$$

and $W^-(\varphi, \psi)$ is decomposed into two parts:

$$W^-(\varphi, \psi) = W_\varphi^-(\psi) + W_\psi^-(\varphi)$$

where

$$W_\varphi^-(\psi) = -\sum_{\xi \in Del_{2,\alpha}^{\alpha_0,-}(\psi, \varphi)} \phi(\xi)$$

and

$$W_{\psi}^{-}(\varphi) = - \sum_{\xi \in Del_{2,\alpha}^{\alpha_0,-}(\psi,\varphi)} \phi(\xi)$$

Proposition 1. The Delaunay model based on the finite energy V is lower regular. Indeed, there exists a positive decreasing integrable function ϕ^* such that

$$W(\varphi, \psi) \geqslant - \sum_{r \in \mathscr{R}} \sum_{s \in \mathscr{S}} \phi^*(|r-s|)\, n(\varphi, r)\, n(\psi, s)$$

where the function ϕ^* is of the form

$$\phi^*(n) = 5 \sup_{t \in [(n-1)\lambda;\, +\infty[} (\tilde{\phi}(t \sin(\alpha_0))) = 5\tilde{\phi}((n-1)\,\lambda \sin(\alpha_0))$$

The rest of this section is devoted to the proof of Proposition 1 decomposed as follows:

- First, we describe the residual edges of the "negative" contribution $W_{\varphi}^{-}(\psi)$.
- Then, we have to associate them with edges of $\varphi \vee \psi$. In fact, the association occurs only between residual edges of $W_{\varphi}^{-}(\psi)$ and "positive" residual edges of some suitable local energy. Therefore, these "negative" edges are decomposed into two parts where for each part one association is proposed. In particular, these associations allow us to control the length and the number of the "negative" edges with respect to "positive" ones.
- Finally, we obtain lower bound of the mutual energy leading to the lower regularity property.

We first decompose $Del_{2,\alpha}^{\alpha_0,-}(\psi, \varphi)$:

$$Del_{2,\alpha}^{\alpha_0,-}(\psi,\varphi) = \bigcup_{\omega \in \psi} \left(B_{\omega}^{0}(\varphi) \cup B_{\omega}^{1}(\varphi) \cup \bigcup_{\omega' \in \psi \setminus \{\omega\}} B_{\omega,\omega'}^{2}(\varphi) \right)$$

with

$$\begin{aligned}
B_{\omega}^{0}(\varphi) &= Del_{2,\alpha}^{\alpha_0,-}(\omega,\varphi) \setminus Del_{2}^{-}(\omega,\varphi) \\
&= Del_{2,\alpha}^{\alpha_0}(\varphi) \cap Del_2(\varphi \cup \{\omega\}) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi \cup \{\omega\})} \\
B_{\omega}^{1}(\varphi) &= Del_{2,\alpha}^{\alpha_0,-}(\omega,\varphi) \cap Del_{2}^{-}(\omega,\varphi) \\
B_{\omega,\omega'}^{2}(\varphi) &= Del_{2,\alpha}^{\alpha_0}(\varphi) \cap Del_{2,\alpha}^{\alpha_0}(\varphi \cup \{\omega\}) \\
&\quad \cap Del_{2,\alpha}^{\alpha_0}(\varphi \cup \{\omega'\}) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi \cup \{\omega,\omega'\})}
\end{aligned}$$

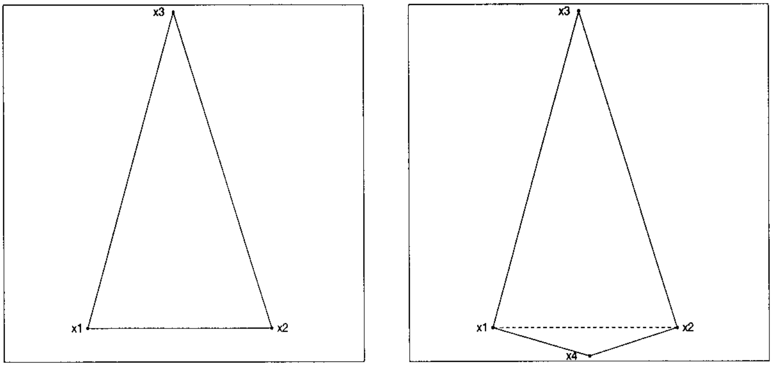

Fig. 2. Since $\gamma(x_4, \{x_1, x_2\}) > \alpha_0$, the edge $\{x_1, x_2\} \in B^0_{x_4}(\{x_1, x_2, x_3\})$.

$B^0_\omega(\varphi)$ corresponds to the set of residual edges that are not broken in the Delaunay graph (see Fig. 2) whereas $B^1_\omega(\varphi)$ and $B^2_{\omega,\omega'}(\varphi)$ are respectively the set of edges that are really broken in the Delaunay graph by the insertion of one point ω of ψ and two points ω and ω' of ψ into the configuration φ (see Fig. 3).

We should notice that at least one of the two points ω or ω', denoted here by ω, satisfies the relation $\gamma(\omega, \xi) \geqslant \pi/2$. Then, we define:

$$B^2_\omega(\varphi) = \bigcup_{\omega' \in \psi \setminus \{\omega\}} \left(B^2_{\omega,\omega'}(\varphi) \cap \left\{ \xi \in \mathscr{P}_2(\varphi) : \gamma(\omega, \xi) \geqslant \frac{\pi}{2} \right\} \right)$$

Now, we are interested in two types of "negative" edges for which we propose two associations with "positive" edges.

The two types of edges (see Fig. 4) are defined, for any $\omega \in \psi$, by:

$$I_\omega(\varphi) = \left(B^1_\omega(\varphi) \cap \left\{ \xi \in Del_2(\varphi) : \gamma(\omega, \xi) \geqslant \frac{\pi}{2} \right\} \right) \cup B^2_\omega(\varphi) \cup B^0_\omega(\varphi)$$

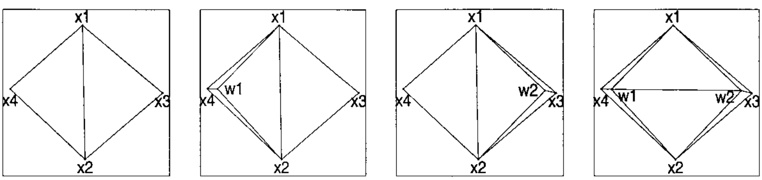

Fig. 3. $\varphi = \{x_1, x_2, x_3, x_4\}$, $\{x_1, x_2\} \in Del_2^{\alpha_0}(\varphi)$, $\{x_1, x_2\} \in Del_2^{\alpha_0}(\varphi \cup \{\omega_1\})$, $\{x_1, x_2\} \in Del_2^{\alpha_0}(\varphi \cup \{\omega_2\})$ but $\{x_1, x_2\} \notin Del_2^{\alpha_0}(\varphi \cup \{\omega_1, \omega_2\})$. Thus, $\{x_1, x_2\} \in B^2_{\omega_1,\omega_2}(\varphi)$.

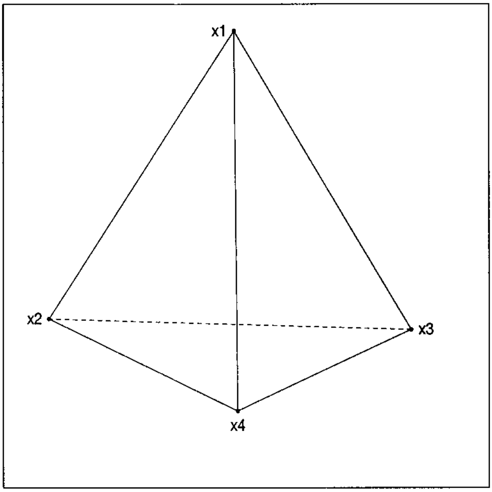

Fig. 4. When $\varphi = \{x_1, x_2, x_3\}$ and $\omega = x_4$, $\{x_2, x_3\} \in I_\omega(\varphi)$ whereas, when $\varphi = \{x_2, x_3, x_4\}$ and $\omega = x_1$, $\{x_2, x_3\} \in II_\omega(\varphi)$.

and

$$II_\omega(\varphi) = B^1_\omega(\varphi) \cap \left\{ \xi \in Del_2(\varphi) : \gamma(\omega, \xi) < \frac{\pi}{2} \right\}$$

Since $\alpha_0 \geqslant \pi/2$, we notice that, for any edge $\xi \in I_\omega(\varphi)$, $\gamma(\omega, \xi) \geqslant \pi/2$.

In order to deal with the association of edges in $I_\omega(\varphi)$, we recall some properties of he Delaunay graph. We then introduce:

$$Del_2^{ext}(\omega, \varphi) = \{ \xi \in Del_2(\varphi \cup \{\omega\}) : \omega \in X^{opp}(\xi, \varphi \cup \{\omega\}) \}$$

$$Del_2^*(\omega, \varphi) = Del_2^-(\omega, \varphi) \cup Del_2^{ext}(\omega, \varphi)$$

Lemma 2. We have the following,

$$N(\omega \mid \varphi) = N_0(\omega \mid \varphi) \cup \bigcup_{\xi \in G_0(\omega \mid \varphi)} N_\xi(\omega \mid \varphi)$$

and

$$Del_2^*(\omega, \varphi) = \bigcup_{\xi \in G_0(\omega \mid \varphi)} G_\xi(\omega \mid \varphi)$$

where

$$N_0(\omega \mid \varphi) = \left\{ z \in N(\omega \mid \varphi) : \left| \,]\omega, z[\cap \bigcap_{\{y_1, y_2\} \in Del_2^-(\omega, \varphi)}]y_1, y_2[\, \right| = 0 \right\}$$

$$G_0(\omega \mid \varphi) = (N_0(\omega \mid \varphi) \vee N_0(\omega \mid \varphi)) \cap Del_2^*(\omega, \varphi)$$

and for any $\xi = \{y_1, y_2\} \in G_0(\omega \mid \varphi)$,

$$N_\xi(\omega \mid \varphi) = \{z \in N(\omega \mid \varphi) \backslash N_0(\omega \mid \varphi) : | \,]\omega, z[\cap]y_1, y_2[\, | \neq 0\}$$

$$G_\xi(\omega \mid \varphi) = ((N_\xi(\omega \mid \varphi) \vee (N_\xi(\omega \mid \varphi) \cup N_0(\omega \mid \varphi))) \cap Del_2^*(\omega, \varphi)) \cup \{\xi\}$$

Notice that for any $\xi_1, \xi_2 \in G_0(\omega \mid \varphi)$ such that $\xi_1 \neq \xi_2$ we have:

$$N_{\xi_1}(\omega \mid \varphi) \cap N_{\xi_2}(\omega \mid \varphi) = \varnothing \qquad \text{and} \qquad G_{\xi_1}(\omega \mid \varphi) \cap G_{\xi_2}(\omega \mid \varphi) = \varnothing$$

and, in particular:

$$(N_{\xi_1}(\omega \mid \varphi) \vee N_{\xi_2}(\omega \mid \varphi)) \cap Del_2^*(\omega, \varphi) = \varnothing \tag{8}$$

Combining this property with some elementary geometrical results, we are able to characterize the sets $cp(\omega, \varphi)$ of subgraphs of $I_\omega(\varphi) \subset Del_2^*(\omega, \varphi)$ describing closed paths (i.e., every vertices have exactly two neighbours). We state:

Lemma 3. $|cp(\omega, \varphi)| \leqslant 1$.

Proof. We prove that if there exists an element of $cp(\omega, \varphi)$, then the point ω is inside it.

- Assume that there exists $L \in cp(\omega, \varphi)$ such that ω is not in the interior of L. Thus, a direct consequence of (8) is the existence of $\xi \in G_0(\omega, \varphi)$ such that $L \subset G_\xi(\omega, \varphi)$. In particular, this means that vertices of L are in $\Pi^{ext}(\omega, \xi)$. But, by definition of $I_\omega(\varphi)$, we have $L \subset \{\xi' \in \mathscr{P}_2(\varphi) : \gamma(\omega, \xi') \geqslant \pi/2\}$ which yields a contradiction.
- Now, assume that there exists at least $L, L' \in cp(\omega, \varphi)$ with $L \neq L'$ for which ω is inside them. As a consequence of (8), L and L' must have the three points of $N_0(\omega \mid \varphi)$ as vertices. Thus, there exists $L'' \in cp(\omega, \varphi)$ satisfying $L'' \subset L \cup L'$ and for which ω is not in the interior of L''. Thus, the proof is complete. ∎

Furthermore, same arguments show that the only possible element of $cp(\omega, \varphi)$ is the set of edges of the triangle $G_0(\omega \mid \varphi)$.

By Lemma 3, it is not difficult to associate each edge of $I_\omega(\varphi)$ to a unique edge of $\varphi \vee \psi$ with length lower than the previous one. More precisely, there exists an injection (not necessarily unique) f_ω^I from $I_\omega(\varphi)$ into $\varphi \vee \psi$ such that:

$$\forall \xi = \{y_1, y_2\} \in I_\omega(\varphi), \qquad f_\omega^I(\xi) = \{\omega, y_1\} \qquad \text{or} \quad f_\omega^I(\xi) = \{\omega, y_2\}$$

Indeed, we may proceed as follows:

- Initially, consider the subgraph $I_\omega(\varphi)$ with vertices denoted by $y_1, \ldots, y_n$.
- Inductively, choose any vertex y_j with exactly one neighbour y_k and associate the edge $\{y_j, y_k\}$ to the edge $\{\omega, y_j\}$ (i.e., $f_\omega^I(\{y_j, y_k\}) = \{\omega, y_j\}$). Next, consider the new subgraph obtained from the previous one without using the edge $\{y_j, y_k\}$ already considered.
- At the end of this procedure, it remains possible to consider the unique closed path for which the association is direct.

Now, we only need to deal with the association of edges in $II_\omega(\varphi)$. Any edge ξ in $II_\omega(\varphi)$ can be associated to the edge $\{\omega, z\}$ where z is the point of $X^{opp}(\xi, \varphi)$ satisfying $\omega \in \mathscr{C}^{ext}(z, \xi)$. Furthermore, by Lemma 1,

$$\|\xi\| \geqslant \|\{\omega - z\}\| \times \sin(\alpha_0)$$

Thus, the application f_ω^{II} defined from $II_\omega(\varphi)$ into $\varphi \vee \psi$ such that:

$$\forall \xi = \{y_1, y_2\} \in II_\omega(\varphi), \qquad f_\omega^{II}(\xi) = \{\omega, z\}$$

is an injection too.

Let us denote

$$I_\psi(\varphi) = \bigcup_{\omega \in \psi} I_\omega(\varphi)$$

$$II_\psi(\varphi) = \bigcup_{\omega \in \psi} II_\omega(\varphi)$$

and notice that

$$Del_{2,\alpha}^{\alpha_0,-}(\psi, \varphi) = I_\psi(\varphi) \cup II_\psi(\varphi)$$

Thus, we define the application $a_{\psi,\varphi}^I$ (resp. $a_{\psi,\varphi}^{II}$) from $\varphi \vee \psi$ into $I_\psi(\varphi) \cup \{\varnothing\}$ (resp. $II_\psi(\varphi) \cup \{\varnothing\}$) by:

$$a_{\psi,\varphi}^I(\{z, \omega\}) = \begin{cases} \xi & \text{if} \quad z = f_\omega^I(\xi) \\ \varnothing & \text{otherwise} \end{cases}$$

$$\left(\text{resp. } a_{\psi,\varphi}^{II}(\{z, \omega\}) = \begin{cases} \xi & \text{if} \quad z = f_\omega^{II}(\xi) \\ \varnothing & \text{otherwise} \end{cases}\right)$$

Hence, the different contributions of the mutual energy become:

$$\begin{aligned} W^{+}(\varphi,\psi) &\geqslant \sum_{\substack{\{z,\omega\}\in Del_{2,\alpha}^{\alpha_0}(\varphi\cup\psi)\\ z\in\varphi,\,\omega\in\psi}} \phi^{-}(\{\omega,z\}) \\ &\geqslant -\sum_{\substack{\{z,\omega\}\in Del_{2,\alpha}^{\alpha_0}(\varphi\cup\psi)\\ z\in\varphi,\,\omega\in\psi}} \tilde{\phi}(\|\omega-z\|) \\ &\geqslant -\sum_{\{z,\omega\}\in\varphi\vee\psi} \tilde{\phi}(\|\omega-z\|\sin(\alpha_0)) \end{aligned}$$

and

$$\begin{aligned} W_{\varphi}^{-}(\psi) &\geqslant -\sum_{\{y_1,y_2\}\in Del_{2,\alpha}^{\alpha_0,-}(\psi,\varphi)} \phi(\{y_1,y_2\}) \\ &\geqslant -\sum_{\{y_1,y_2\}\in I_{\psi}(\varphi)} \tilde{\phi}(\|y_1-y_2\|) - \sum_{\{y_1,y_2\}\in II_{\psi}(\varphi)} \tilde{\phi}(\|y_1-y_2\|) \\ &\geqslant -\sum_{\{z,\omega\}\in(a_{\psi,\varphi}^{I})^{-1}(I_{\psi}(\varphi))} \tilde{\phi}(\|\omega-z\|\sin(\alpha_0)) \\ &\quad -\sum_{\{z,\omega\}\in(a_{\psi,\varphi}^{II})^{-1}(II_{\psi}(\varphi))} \tilde{\phi}(\|\omega-z\|\sin(\alpha_0)) \\ &\geqslant -2\sum_{\{z,\omega\}\in\varphi\vee\psi} \tilde{\phi}(\|\omega-z\|\sin(\alpha_0)) \end{aligned}$$

as a consequence of Lemma 1. By symmetry, we obtain the same inequality for $W_{\psi}^{-}(\varphi)$.

Finally, it follows

$$\begin{aligned} W(\varphi,\psi) &\geqslant -5\sum_{\{z,\omega\}\in\varphi\vee\psi} \tilde{\phi}(\|\omega-z\|\sin(\alpha_0)) \\ &\geqslant -\sum_{r\in\mathcal{R}}\sum_{s\in\mathcal{S}} \phi^{*}(|r-s|)\,n(\varphi,r)\,n(\psi,s) \end{aligned}$$

which completes the proof.

4.3. Quasilocality on U_m

Given $\Lambda, \tilde{\Lambda}\in\mathcal{B}_b$ such that $\Lambda\subset\tilde{\Lambda}$. $\tilde{\Lambda}$ will increase to $\mathbb{R}^2$ (in the Van Hove sense) in order to obtain thermodynamic limit. Let us choose n_{Λ} such that $C_{n_\Lambda} \supset \Lambda$ and take $n_{\tilde{\Lambda}, \inf}$ the greatest integer and $n_{\tilde{\Lambda}, \sup}$ the lowest integer such that

$$C_{n_{\tilde{\Lambda}, \inf}} \subset \tilde{\Lambda} \subset C_{n_{\tilde{\Lambda}, \sup}}$$

We may assume that $n_{\tilde{\Lambda}, \inf}$ and $n_{\tilde{\Lambda}, \sup}$ upper than n_Λ. Let us decompose any configuration $\varphi \in \Omega_f$ as follows:

$$\varphi = \varphi^i \cup \varphi^b \cup \varphi^o$$

with φ^i the subconfiguration inside Λ, φ^o the subconfiguration outside $\tilde{\Lambda}$ and φ^b the subconfiguration between Λ and $\tilde{\Lambda}$, say $\varphi \cap (\tilde{\Lambda} \setminus \Lambda)$. For the α-Delaunay model, we obtain

$$V_\Lambda(\varphi^i, \varphi) - V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi) = V(\varphi) - V(\varphi^b \cup \varphi^o) - V(\varphi^i \cup \varphi^b) + V(\varphi^b)$$

By defining,

$$\Theta_1 = \{\xi = \{y_1, y_2\} : y_1 \in \varphi^i,\ y_2 \in \varphi^o,\ \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi)\}$$

$$\Theta_2 = \{\xi = \{y_1, y_2\} \subset \varphi^b : \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o)} \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b)}\}$$

$$\Theta_3 = \{\xi = \{y_1, y_2\} \subset \varphi^b : \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o) \cap Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}\}$$

$$\Theta_4^i = \{\xi = \{y_1, y_2\} \subset \varphi^i : \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}\}$$

$$\Theta_5^i = \{\xi = \{y_1, y_2\} : y_1 \in \varphi^i,\ y_2 \in \varphi^b,\ \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}\}$$

$$\Theta_4^o = \{\xi \in \{y_1, y_2\} \subset \varphi^o : \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}\}$$

$$\Theta_5^o = \{\xi = \{y_1, y_2\} : y_1 \in \varphi^o,\ y_2 \in \varphi^b,\ \xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}\}$$

$$\Theta_6^i = \left\{\xi = \{y_1, y_2\} : y_1 \in \varphi^o,\ y_2 \in \varphi^b \cup \varphi^o,\ \xi \in \bigcup_{\omega \in \varphi^i} B_\omega^0(\varphi^b \cup \varphi^o)\right\}$$

$$\Theta_6^o = \left\{\xi = \{y_1, y_2\} : y_1 \in \varphi^i,\ y_2 \in \varphi^i \cup \varphi^b,\ \xi \in \bigcup_{\omega \in \varphi^o} B_\omega^0(\varphi^i \cup \varphi^b)\right\}$$

$$PP = \{\xi = \{y_1, y_2\} : \exists \omega_1 \in \varphi^i, \exists \omega_2 \in \varphi^o : \|y_1 - y_2\| > \sin(\alpha_0)\ \|\omega_1 - \omega_2\|\}$$

$$\Theta^+ = (\Theta_1 \cup \Theta_2) \cap PP$$

$$\Theta^- = \left(\Theta_3 \cup \left(\bigcup_{k=4}^{6} \Theta_k^i\right) \cup \left(\bigcup_{k=4}^{6} \Theta_k^o\right)\right) \cap PP$$

we state the following result.

Proposition 2. 1. For any $\tilde{\Lambda} \supset \Lambda$ and any $\varphi \in \Omega$,

$$V_\Lambda(\varphi^i, \varphi) - V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi) = \sum_{\xi \in \Theta^+} \phi(\xi) - \sum_{\xi \in \Theta^-} \phi(\xi) \tag{9}$$

2. For any $\tilde{\Lambda}$ large enough,

$$|V_\Lambda(\varphi^i, \varphi) - V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)| \leqslant |\varphi^i| \, \delta_m(\tilde{\Lambda}) \qquad \text{uniformly in} \quad \varphi \in U_m$$

where δ_m vanishes asymptotically. Consequently, the Delaunay pairwise model satisfies the condition (3.13).

The proof of this proposition is given in appendix. In particular, the proof of the second part of this proposition relies on some arguments given by Preston when proving the same condition (3.13) for the superstable and regular pairwise interaction model.

5. EXISTENCE OF MIXED PAIRWISE MODEL

Using the Theorem 3.3 of Preston,[15] we proved the existence of Gibbs state for the mixed pairwise model.

Proposition 3. The mixed pairwise model satisfies the conditions (3.7), (3.12) and (3.13) of the Preston's theorem. Then the set of Gibbs states associated to the mixed finite energy is non empty.

Proof. Condition (3.12): On the one hand,

$$(V_1 \text{ Superstable and } V_2 \text{ Stable}) \Rightarrow (V = V_1 + V_2 \text{ Superstable})$$

and, on the other hand,

$$(V_1 \text{ and } V_2 \text{ Lower Regular}) \Rightarrow (V = V_1 + V_2 \text{ Lower Regular})$$

It implies that according to the Ruelle's results and as shown by Preston in the lemma (6.8), V satisfies (3.12).

Condition (3.7): For any $F \in \tilde{\mathscr{F}}_\Lambda$,

$$\pi_\Lambda(\emptyset, F) = \pi_\Lambda(\emptyset, F \cap U_{m_0}) + \pi_\Lambda(\emptyset, F \cap U_{m_0}^c)$$

From (3.12), we know that

$$\pi_\Lambda(\emptyset, F \cap U_{m_0}^c) \leqslant \delta \qquad \text{uniformly on } \Lambda$$

Now,

$$\pi_\Lambda(\emptyset, F\cap U_{m_0}) = \int_{U'_{m_0}} \pi_\Lambda(t, F\cap U_{m_0})\, \pi_\Lambda(\emptyset, dt)$$
$$\leqslant \sup_{t\in U'_{m_0}} \pi_\Lambda(t, F\cap U_{m_0})$$

where $U'_{m_0} = \{\psi\in\Omega_\Lambda : \psi\in U_{m_0}\}$. However, for any $t\in U'_{m_0}$,

$$\pi_\Lambda(t, F\cap U_{m_0}) = \frac{1}{Z_\Lambda(t)} \int_{F\cap U_{m_0}} e^{-V_\Lambda(x\mid t)}\mathbf{Q}_\Lambda(dx)$$
$$\leqslant \int_{F\cap U_{m_0}} e^{D_{m_0}|x|}\mathbf{Q}_\Lambda(dx) = \omega(F)$$

by using Lemma 5. Condition (3.13): (V_1 and V_2 satisfy (3.13)) $\Rightarrow$ ($V = V_1 + V_2$ satisfies (3.13)). ▮

When we replace the superstable component by a hard-core condition on the α-Delaunay pairwise potential, we can establish the local stability and quasilocality properties by using similar arguments as Klein.[11] This leads to the existence of Gibbs State by Theorem 3.1 of Preston.[15]

6. APPENDIX: PROOF OF PROPOSITION 2

6.1. Proof of Proposition 2.1

This proof is to determine every residual edges in the difference between $V_\Lambda(\varphi^i, \varphi)$ and $V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)$. Furthermore, we show that these residual edges are in *PP*.

1. $\xi = \{y_1, y_2\}$ with $y_1\in\varphi^i$, $y_2\in\varphi^o$:

Necessarily, $\xi\in Del^{\alpha_0}_{2,\alpha}(\varphi)\cap\overline{Del^{\alpha_0}_{2,\alpha}(\varphi^b\cup\varphi^o)}\cap\overline{Del^{\alpha_0}_{2,\alpha}(\varphi^i\cup\varphi^b)}\cap\overline{Del^{\alpha_0}_{2,\alpha}(\varphi^b)}$ (i.e., $\xi\in\Theta_1$).

2. $\xi = \{y_1, y_2\}$ with $y_1, y_2\in\varphi^b$:

We have $\xi\notin Del^{\alpha_0}_{2,\alpha}(\varphi)$ since otherwise using relation (1)

$$\xi\in Del^{\alpha_0}_{2,\alpha}(\varphi^b\cup\varphi^o)\cap Del^{\alpha_0}_{2,\alpha}(\varphi^i\cup\varphi^b)\cap Del^{\alpha_0}_{2,\alpha}(\varphi^b)$$

ξ is not residual in the quantity $V_\Lambda(\varphi^i, \varphi^b\cup\varphi^o) - V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi^b\cup\varphi^o)$. Likewise from relation (1), we can prove that the residual edges are:

(i) $\xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o)} \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b)}$: in this case, $\xi \in \Theta_2$. We have to prove that $\xi \in PP$. Let $z \in \varphi^b$ such that $\{y_1, y_2, z\} \in Del_3(\varphi^b)$. Since $\{y_1, y_2, z\} \notin Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b) \cup Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o)$, there exists $\omega_1 \in \varphi^i$ and $\omega_2 \in \varphi^o$ such that $\omega_1, \omega_2 \in \mathscr{C}^{ext}(z, \{y_1, y_2\})$. Under Lemma 1 and relation (6) we conclude that $\xi \in PP$.

(ii) $\xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o) \cap Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b)$: in this case, $\xi \in \Theta_3$. We have to prove that $\xi \in PP$. There exists $z \in \varphi^o$ (resp. $z \in \varphi^i$) such that[2]

$$\{y_1, y_2, z\} \in Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)} \text{ (resp. } \{y_1, y_2, z\} \in Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}$$

Thus, there exists $\omega \in \varphi^i$ (resp. $\omega \in \varphi^o$) such that

$$\omega \in \mathscr{C}^{ext}(z, \{y_1, y_2\})$$

Lemma 1 and relation (6) lead to $\xi \in PP$.

3. $\xi = \{y_1, y_2\}$ with $y_1, y_2 \in \varphi^i$: according to relation (1), the only case to consider is $\xi \in \Theta_4^i$. Thus, there exists $z \in \varphi^i \cup \varphi^b$ and φ^o such that

$$\omega \in \mathscr{C}^{ext}(z, \{y_1, y_2\})$$

Lemma 1 and relation (6) lead to $\xi \in PP$.

4. $\xi = \{y_1, y_2\}$ with $y_1 \in \varphi^i$, $y_2 \in \varphi^b$: according to relation (1), the only case to consider is $\xi \in \Theta_5^i$. Thus, there exists $z \in \varphi^i \cup \varphi^b$ and $\omega \in \varphi^o$ such that

$$\omega \in \mathscr{C}^{ext}(z, \{y_1, y_2\})$$

Lemma 1 and relation (6) lead to $\xi \in PP$.

5. Unfortunately, when $\xi = \{y_1, y_2\}$ with $y_1 \in \varphi^o$ and $y_2 \in \varphi^b \cup \varphi^o$, it may happen that $\xi \in \bigcup_{\omega \in \varphi^i} B_\omega^0(\varphi^b \cup \varphi^o)$, i.e., $\xi \in \Theta_6^i$. Since there exists $z \in \varphi^i$ such that

$$\xi \cup \{z\} \in Del_3(\varphi) \qquad \text{and} \qquad \gamma(z, \{y_1, y_1\}) \geqslant \alpha_0 > \frac{\pi}{2}$$

[2] In the two cases, we exclude the eventuality $z \in \varphi^b$. Indeed, if $z \in \varphi^b$ then $\{y_1, y_2, z\} \in Del_3(\varphi^b)$. Since $\xi \in Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o) \cap Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b) \cap \overline{Del_{2,\alpha}^{\alpha_0}(\varphi)}$ there exists $\omega \in \varphi^o$ or $\omega \in \varphi^i$ such that

$$\omega \in \mathscr{C}^{ext}(z, \{y_1, y_2\})$$

then, $\xi \notin Del_{2,\alpha}^{\alpha_0}(\varphi^i \cup \varphi^b)$ or $\xi \notin Del_{2,\alpha}^{\alpha_0}(\varphi^b \cup \varphi^o)$ which yields a contradiction with our asumption. Thus, $z \notin \varphi^b$.

it follows that

$$\|y_1 - y_2\| \geqslant \|z - y_1\| \geqslant \sin(\alpha_0)\, \|z - y_1\|$$

Thus, $\xi \in PP$.

From the symmetry between φ^i and φ^o in the quantity $V_\Lambda(\varphi^i, \varphi) - V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)$, the cases θ_4^o, Θ_5^o and Θ_6^o are respectively the symmetric cases of Θ_4^i, Θ_5^i, and Θ_6^i. Moreover, the proofs are similar. ∎

6.2. Proof of Proposition 2.2

In order to simplify the notation, we take

$$k_n = \sin(\alpha_0) \times (n - n_\Lambda) \qquad \text{and} \qquad \beta_n = |\Lambda_n|$$

Lemma 4. Given any $\varphi^i \in \Omega_\Lambda$, $\lim_{\tilde{\Lambda} \to \mathbb{R}^2} V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi) = V_\Lambda(\varphi^i, \varphi)$ uniformly on $\varphi \in U_m$.

Proof. Let n_0 be the integer such that for any $\tilde{\Lambda} \supset C_{n_0}$, we have

$$k_{n_{\tilde{\Lambda}, \inf}} \geqslant \operatorname{diam}(\Lambda)$$

Then, we first prove that for any $\tilde{\Lambda} \supset C_{n_0}$

$$|V_\Lambda(\varphi^i, \varphi) - V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)| \leqslant |\varphi^i|\, \delta_m(\tilde{\Lambda}) \qquad \text{uniformly in} \quad \varphi \in U_m$$

where

$$\delta_m(\tilde{\Lambda}) = 4 \sum_{n = n_{\tilde{\Lambda}, \inf}}^{+\infty} m\beta_n \tilde{\phi}(k_n) + 10 m n_{\tilde{\Lambda}, \sup}^2 \tilde{\phi}(k_{n_{\tilde{\Lambda}, \inf}})$$

Indeed, in the proof of Proposition 2.1, we showed that each residual edge in the difference between $V_\Lambda(\varphi^i, \varphi)$ and $V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)$ is in PP. Now, we can prove that the number and the length of these residual edges are controlled. More precisely, it follows:

1. $\xi \in \Theta_1$: if $y_1 \in \varphi^i$ and $y_2 \in \Lambda_n$ then there exists no more than $2(|\varphi^i| + m\beta_n) \leqslant 2m\beta_n\, |\varphi^i|$ edges ξ with length greater than k_n.
2. $\xi \in \Theta_2$: there exists no more than $3mn_{\tilde{\Lambda}, \sup}^2 \leqslant 3mn_{\tilde{\Lambda}, \sup}^2\, |\varphi^i|$ edges ξ with length $k_{n_{\tilde{\Lambda}, \inf}}$.
3. $\xi \in \Theta_3$: similar to Θ_2.
4. $\xi \in \Theta_4^i$: since $k_{n_{\tilde{\Lambda}, \inf}} \geqslant \operatorname{diam}(\Lambda)$, such type of edges do not exist when $\tilde{\Lambda}$ is large enough.

5. $\xi \in \Theta_5^i$: there exists no more than $2(|\varphi^i| + mn^2_{\tilde{\Lambda}, \sup}) \leqslant 2mn^2_{\tilde{\Lambda}, \sup} |\varphi^i|$ edges ξ with length $k_{n_{\tilde{\Lambda}, \inf}}$.

6. $\xi \in \Theta_6^o$: similar to Θ_4^i.

7. $\xi \in \Theta_4^o \cup \Theta_5^o$: there exists $z \in \varphi^b \cup \varphi^o$, $\omega_1, \omega_2 \in \varphi^i$ such that

$$\{y_1, z, \omega_1\} \in Del_3(\varphi) \qquad \text{and} \qquad \{y_2, z, \omega_2\} \in Del_3(\varphi)$$

and we may associate in a unique way the edge $\{z, \omega_1\} \in Del_2(\varphi)$ to the edge $\{y_1, y_2\}$. Hence, if $z \in \varphi^b$ the number of edges ξ is lower than $2(|\varphi^i| + mn^2_{\tilde{\Lambda}, \sup}) \leqslant 2mn^2_{\tilde{\Lambda}, \sup} |\varphi^i|$, and, if $z \in \varphi^o$ the number of edges ξ is lower than number of edges of Θ_1.

8. $\xi \in \Theta_6^i$: we may associate in a unique way the edge $\{z, y_2\} \in Del_2(\varphi)$ to the edge $\{y_1, y_2\}$ where $z \in \varphi^i$ is such that $\xi \in B_z^0(\varphi^b \cup \varphi^o)$. ▮

Let us notice that a consequence of

$$\lim_{\tilde{\Lambda} \to \mathbb{R}^2} \delta_m(\tilde{\Lambda}) = 0$$

is the existence of $n_1 > 0$ and $\delta_m > 0$ such that, for any $\tilde{\Lambda} \supset C_{n_1}$,

$$\delta_m(\tilde{\Lambda}) \leqslant \delta_m < +\infty$$

Lemma 5. There exists some positive constant D_m such that

$$V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi) \geqslant -D_m |\varphi^i|$$

uniformly on $\varphi \in U_m$ and on $\tilde{\Lambda} \supset C_{n_1}$.

The proof of this lemma is more or less proposed by Ruelle,[17] p. 140 as a consequence of the lower regularity property.

Proof of Condition (3.13) for the Delaunay Pairwise Model. Similarly to Preston in the Lemma 6.7 (p. 106 and 107), we only need to prove the convergence for any $G \in \tilde{\mathscr{F}}_\Lambda$ of $\int_G e^{-V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)} \mathbf{Q}_\Lambda(d\varphi^i)$ to $\int_G e^{-V_\Lambda(\varphi^i, \varphi)} \mathbf{Q}_\Lambda(d\varphi^i)$ uniformly on $\varphi \in U_m$. From the previous Lemma 5, we have

$$\begin{aligned}
&|e^{-V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi)} - e^{-V_\Lambda(\varphi^i, \varphi)}| \\
&\qquad \leqslant |V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi) - V_\Lambda(\varphi^i, \varphi)| \, e^{\max\{-V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi), -V_\Lambda(\varphi^i, \varphi)\}} \\
&\qquad \leqslant |V_\Lambda^{\tilde{\Lambda}}(\varphi^i, \varphi) - V_\Lambda(\varphi^i, \varphi)| \, e^{D_m |\varphi^i|}
\end{aligned}$$

It follows from Lemma 4 that for any $\tilde{\Lambda} \supset C_{n_1}$,

$$|e^{-V_{\Lambda}^{\tilde{\Lambda}}(\varphi^i, \varphi)} - e^{-V_{\Lambda}(\varphi^i, \varphi)}| \leqslant \delta_m(\tilde{\Lambda}) |\, \varphi^i |\, e^{D_m |\varphi^i|} \leqslant \delta_m \, |\varphi^i| \, e^{D_m |\varphi^i|}$$

Since

$$\int_G |\varphi^i| \, e^{D_m |\varphi^i|} \mathbf{Q}_{\Lambda}(d\varphi^i) = \sum_{n \geqslant 1} n \, \frac{(\lambda e^{D_m} |\Lambda|)^n}{n!} = \lambda e^{D_m} |\Lambda| \exp(\lambda e^{D_m} |\Lambda|) < +\infty$$

we can apply the Lebesgue's dominated convergence theorem. ∎

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